JOURNAL OF APPROXIMATION THEORY 18, 136–151 (1976)

On Rational L_2-Approximation

DIETRICH BRAESS

Institut für Mathematik, Ruhr-Universität Bochum, 463 Bochum, West Germany

Communicated by E. W. Cheney

Received March 26, 1975

Es wird gezeigt, daß es bei der rationalen L_2-Approximation über einem Intervall keine universelle Schranke für die Zahl der lokalen Minima gibt. Für spezielle Zähler- und Nennergrade wurde dies bereits von Wolfe bewiesen. Die diskrete Approximation wird dagegen abgesetzt. Schärfere Aussagen ergeben sich auch beim Nennergrad eins, womit Ergebnisse von Spieß vervollständigt werden, sowie für den Nennergrad zwei.

1. INTRODUCTION

This paper is concerned with mean-square approximation in the family of rational functions

$$R_{l,r} = \{g = p/q;\ \partial p \leqslant l,\ \partial q \leqslant r,\ q(t) > 0 \text{ for } t \in [-1, +1]\}.$$

(Here ∂p and ∂q denote the degrees of the polynomials p and q, resp.) According to the general theory of Efimov and Stechkin [10] and of Vlasov [18], one cannot expect that the best approximation is always unique (cf. also [1, p. 178]). Indeed, examples of functions with two best approximations were constructed by several authors [9, 12, 16] using a symmetry argument [13] and the nondegeneracy result of Cheney and Goldstein [8].

On the other hand, Wolfe has shown in a recent paper [19] that uniqueness is a generic property [15, p. 18], i.e., uniqueness holds for a dense open subset. Therefore, at first glance nonuniqueness does not seem to be a problem from the numerical point of view, because rounding errors will almost always cause uniqueness. But the existence of more than one best approximation in a few cases implies the existence of more than one local best approximation (lba) in many cases. Uniqueness of an lba is not a generic property. When iterative methods are applied for the computation of the optimum, then unfortunately the lba's may hinder the algorithm from finding the global solution.

How can one estimate the number of lba's? The first step in this direction was the result of Wolfe, that there is no uniform bound on the number of local best approximations in $R_{l,r}$ $l \leqslant r - 1$. This result will be extended to the case $l \geqslant r$. On the other hand, we will prove that Wolfe's result is sharp: It is true only when the underlying set of the approximation problem is an interval, but not if it is a finite point set.

Now the question arises whether the number of lba's is at least finite. In [19, Remark 5] finiteness was reported for the case $r = 1$, i.e., for $R_{l,1}$, but it turned out that this information was based on a (linguistic) translation error. Now, a proof will be established under certain boundary conditions.

To treat the case $r = 2$, new methods must be developed. The main idea stems from the theory of minimal surfaces. It is shown that the critical points are either isolated or belong to one-dimensional analytical manifolds. Moreover, tools which are used up to now "ad hoc," are put in a general framework (which also is not completely new).

There remain many unsolved questions, but we hope that the reader will consider this more as a stimulus than as a drawback. Moreover, the techniques used are not restricted to the space $L_2[-1, +1]$, and the reader may extend the results to approximation problems in those Hilbert spaces which are investigated in connection with optimal quadrature formulas [2].

2. Critical Points, Degeneracy

Though our aim is the investigation of rational functions, we will give the basic notation in a more abstract framework. In this way we obtain a larger independency of the parametrization of the family $R_{l,r}$. Moreover, we do not yet restrict ourselves to the L_2-norm but consider the approximation in a real space H with an inner product $[\cdot, \cdot]$. The reader is referred to [14, pp. 32–38] for more details.

Let A be an open set in n-space. A continuous mapping $F: A \to H$ defines the approximating family

$$G = \{F(a); a \in A\} \subset H.$$

Assume that the first and the second derivatives d_aF and $d_a{}^2F$ in the sense of Fréchet exist. Here d_aF is a linear transformation: $\mathbb{R}^n \to H$ and $d_a{}^2F$ is a bilinear symmetric form on n-space: $\mathbb{R}^n \times \mathbb{R}^n \to H$. If the kernel ker d_aF consists only of zero, then the tangent space at $g = F(a)$ is given by

$$T_gG = \{d_aF \cdot b; b \in \mathbb{R}^n\} = d_aF(\mathbb{R}^n), \tag{2.1}$$

at least after reducing the parameter set A, if necessary [3].

The analysis is based on the square of the distance function,

$$\rho(a) = \|f - F(a)\|^2 = [f - F(a), f - F(a)]. \tag{2.2}$$

An element $g = F(a)$ is a best approximation (local best approximation, resp.) to f in G, if ρ has a minimum (a local minimum, resp.) at the point a. The derivatives of ρ are easily derived.

$$d_a\rho = -2[f - F(a), d_aF], \tag{2.3}$$

$$\tfrac{1}{2}d_a{}^2\rho = [d_aF, d_aF] - [f - F(a), d_a{}^2F], \tag{2.4}$$

which may be considered as abbreviations of

$$d_a\rho \cdot b = -2[f - F(a), d_aF \cdot b],$$

$$\tfrac{1}{2}d_a{}^2\rho \cdot b_1 \cdot b_2 = [d_aF \cdot b_1, d_aF \cdot b_2] - [f - F(a), d_a{}^2F \cdot b_1 \cdot b_2].$$

An element $g = F(a)$ is called a *critical point* if $d_a\rho = 0$. Obviously, each lba is a critical point. The converse is not true in general. For this reason second-order terms are analyzed.

DEFINITION 2.1. Let $F(a)$ be a critical point. The number of negative eigenvalues of $d_a{}^2\rho$ is called its *index*, and the dimension of ker $d_a{}^2\rho$ is called its *nullity*. A critical point is *degenerate* if its nullity is greater than 0.

The index and the nullity do not change if another parameterization is chosen which is related to the given one through a C^2-mapping. Thus, it is not necessary to distinguish between an element $g \in G$ and its parameter and we will sometimes call a instead of $F(a)$ a critical point.

Throughout this paper the term degeneracy is used in the sense of Definition 2.1. The complication which arises from degeneracy becomes apparent from the following lemma.

LEMMA 2.1. *Nondegenerate critical points are isolated.*

Proof. Put

$$\phi_i(a) = (\partial/\partial a_i)\, \rho(a), \qquad i = 1, 2,..., n. \tag{2.5}$$

If a_0 is a nondegenerate critical point, then

$$\det \left| \frac{\partial}{\partial a_k} \phi_i(a_0) \right|_{i,k=1}^{n} \neq 0.$$

It follows from the implicit function theorem that in a sufficiently small neighborhood of a_0 there is exactly one solution of

$$\phi_i(a) = y_i, \qquad i = 1, 2,..., n, \tag{2.6}$$

provided that y_i is sufficiently small. Hence, a_0 is an isolated solution of (2.6) with $y_1 = y_2 = \cdots = y_n = 0$. $\square$

Note that the classical criteria say the following. If a critical point is an lba, then its index is zero. On the other hand, each critical point with vanishing index and nullity corresponds to a (strict) lba.

Expression (2.4) splits into two terms in a natural way. As usual, they are called the first and the second fundamental form, respectively [14, p. 33]. The first fundamental form is positive definite, provided that $\ker d_aF = \{0\}$.

Let a be a critical point to f in G. Then this point a is also critical when the functions

$$f_\lambda = F(a) + \lambda(f - F(a)), \qquad \lambda > 0,$$

are approximated. The corresponding second derivative is

$$[d_aF, d_aF] - \lambda[f - F(a), d_a^2F]. \tag{2.7}$$

Hence, only the second fundamental form depends on λ. The index vanishes for sufficiently small λ. This implies uniqueness whenever f is sufficiently close to the approximating family [17, 19].

3. Nonexistence of a Uniform Bound

Let the space $L_2[-1, +1]$ be endowed with the inner product

$$[f, g] = \int_{-1}^{+1} f(t)\, g(t)\, dt. \tag{3.1}$$

As was pointed out by Wolfe, there is no bound on the number of lba's in $R_{l,r}$ which is independent of f, if $l \leqslant r - 1$. This is a consequence of [19, Theorem 6], which is given for the reader's convenience.[1]

THEOREM 3.1. *Let $l < r$. Assume that $g_i = p_i/q_i \in R_{l,r} \backslash R_{l-1,r-1}$, $\partial q_i = r$, $i = 1, 2,..., N$, are such that q_i and q_j have no common factors unless $i = j$. Then there is an $f \in L_2[-1, +1]$ to which $g_1, g_2,..., g_N$ are local best approximations in $R_{l,r}$.*

It is the aim of this section to present an extension to the case $l \geqslant r$.

THEOREM 3.2. *Let $l \geqslant r \geqslant 1$, and put $m = l - r$. Assume that $p_i/q_i \in R_{r-1,r} \backslash R_{r-2,r-1}$, $\partial q_i = r$, $i = 1, 2,..., N$, are such that q_i and q_j have no common factors unless $i = j$. Then there are polynomials u_i, $\partial u_i \leqslant m$, $i = 1, 2,..., N$ and there is an $f \in L_2[-1, +1]$ such that each $g_i = u_i + p_i/q_i$ is a local best approximation to f in $R_{l,r}$.*

Remark. Theorem 3.1 cannot be extended to the case $l \geqslant r$ without modification, namely, if f_i is an lba to f, then zero is the best approximation to $f - g_i$ in $P_m := R_{m,0}$. Hence, $f - g_i$ is contained in $P_m^\perp$, the orthogonal complement of P_m. This implies $g_i - g_j \in P_m^\perp$ for all i, j.

[1] The zero polynomial has the degree -1.

To overcome this difficulty the polynomials u_i, $i = 1, 2, \ldots, N$ will be chosen as the best approximations of $(-p_i/q_i)$ in p_m. Consequently, $g_i \in P_m{}^\perp$.

LEMMA 3.3. *Assume that for* $q_1, q_2, \ldots, q_N$ *the conditions of Theorem* 3.2 *prevail. Then* $h_0 \in P_m$, $h_i \in P_{3r-1}$, $i = 1, 2, \ldots, r$ *and*

$$h_0 + \sum_{i=1}^{N} (h_i/q_i{}^3) = 0 \tag{3.2}$$

imply $h_0 = h_1 = \cdots = h_N = 0$.

Proof. After multiplying (3.2) by $q_j{}^3$, we obtain

$$h_j = -q_j{}^3 \left\{ \sum_{i \neq j} h_i/q_i{}^3 + h_0 \right\}.$$

Since the degree of q_j is exactly r, the polynomial h_j has at least $3r$ zeros counting multiplicities. Hence, $h_j = 0$ for $j = 1, 2, \ldots, N$. Now, $h_0 = 0$ is immediate. □

Proof of Theorem 3.2. The main idea is to show the existence of an element f, such that

$$[f - g_i, d_{a_i} F] = 0, \tag{3.3}$$

$$[f - g_i, d_{a_i}^2 F] = 0, \tag{3.4}$$

where $g_1, g_2, \ldots, g_N$ are constructed as specified in the remark above. Note that (3.4) means that the second fundamental form vanishes and that $d^2\rho$ coincides with the first fundamental form. This implies that g_i is a critical point with vanishing index and nullity. Hence, each g_i is an lba.

The rational functions in $R_{l,r}$ may be written as the sum of functions in $R_{r-1,r}$ and of P_m. Consequently, the tangent space at g_i is spanned by the tangent spaces of $R_{r-1,r}$ at p_i/q_i and by the tangent space of P_m at u_i. The tangent space of P_m coincides with P_m and the other tangent space was calculated in [19]; it is $q_i^{-2} P_{2r-1}$. Reformulating (3.3) as

$$f - g_i \in (T_{g_i} R_{l,r})^\perp$$

leads to the equivalent relations

$$\begin{aligned} f - g_i &\in (q_i^{-2} \cdot P_{2r-1})^\perp, \\ f - g_i &\in P_m{}^\perp. \end{aligned} \tag{3.5}$$

Since by construction $g_i \in P_m{}^\perp$, the latter relation is reduced to

$$f - 0 \in P_m. \tag{3.6}$$

When calculating second derivatives we may neglect the linear space P_m and restrict our attention to $R_{r-1,r}$. As was shown in [19], the functions $\{d^2_{a_i}F \cdot b \cdot b; b \in \mathbb{R}^2{}_\alpha\}$ corresponding to $R_{r-1,r}$ are contained in $q_i^{-3} \cdot P_{3r-1}$. Hence, the relations (3.4) are guaranteed, if we can establish

$$f - g_i \in (q_i^{-3} \cdot P_{3r-1})^{\perp}, \qquad i = 1, 2,..., N. \tag{3.7}$$

Obviously, (3.7) implies (3.5). Now, combining [19, Lemma 6] and Lemma 3.3 it follows that there is indeed a function f satisfying (3.6) and (3.7). This completes the proof. □

Actually, the rational functions g_i are strictly local best approximations to the constructed function f, because the bilinear forms $d^2_{a_i}\rho$ are positive definite.

DEFINITION 3.1. g is called a *strictly local best approximation* to f in G, if it is the unique best approximation in some open neighborhood U of g in G.

For strict best approximations the following lemma holds, the proof of which is omitted.

LEMMA 3.4. *Let g_0 be a strict local best approximation to f_0 in a locally compact set G. Then there is a neighborhood U of g_0 in G and a $\delta > 0$, such that there is at least one local best approximation to f in U, whenever $\|f - f_0\| < \delta$.*

Let $g_1, g_2, ..., g_N$ be defined as in Theorem 3.2 and let f_0 be a function as constructed in the proof. Since $R_{l,r} \backslash R_{l-1,r-1}$ is an $(l + r + 1)$-dimensional manifold, it is locally compact and Lemma 3.4 may be applied N times. We obtain at least N l.b.a.'s for all functions sufficiently close to f_0. Hence, we obtain as a consequence

COROLLARY 3.5. *Uniqueness of local best approximations in $R_{l,r}$ is not a generic property.*

We conclude this section with two problems.

PROBLEM 3.1. In the theory of optimal quadrature formulas $f - g$ is called a monospline [4, 7] if $g \in R_{l,r}$ and f possesses a representation

$$f(t) = u(t) + \int_{-1}^{+1} (t^{l-r+1}/(1 - xt))\, d\mu(x), \tag{3.8}$$

where $\partial u \leqslant l - r$ and $d\mu$ is a nonnegative measure. The functions of the form (3.8) define a cone. Can Wolfe's construction be extended to establish functions f in this cone with a given number of local best approximation?

PROBLEM 3.2. Is it possible to exhibit a function f which has at least 3 best approximations (not only with 3 local best approximations)?

4. THE CASE $r = 1$

The rational functions in $R_{l,1}$ may be represented in the form

$$F(a, t) = (\alpha t^l/(1 - xt)) + \sum_{\mu=0}^{l-1} \beta_\mu t^\mu, \tag{4.1}$$

where the parameter $a = (\alpha, x, \beta_0, ..., \beta_{l-1})$ is an element in $(l + 2)$-space. If the approximation is taken on the interval $-1 \leqslant t \leqslant +1$, then the characteristic number x is restricted to $(-1, +1)$, and

$$A = \{(\alpha, x, \beta_0, ..., \beta_{l-1}) \in \mathbb{R}^{l+2};\ -1 < x < 1\}.$$

Moreover, we adopt the convention that the sum in (4.1) is void if $l = 0$.

If the parameter x is fixed, then the approximation problem is reduced to a linear problem and optimal values for $\alpha, \beta_0, ..., \beta_{l-1}$ are associated to x. Thus, a one-dimensional manifold is defined. It will turn out that this is even an analytical submanifold of $L_2[-1, +1]$. Obviously, each critical point is contained in this manifold. But which of its elements are really critical points?

Let $g = F(a)$, $\alpha \neq 0$. By computing derivatives we obtain

$$T_g R_{l,1} = \operatorname{span}\{1, t, ..., t^{l-1}, t^l(1 - xt)^{-1}, t^{l+1}(1 - xt)^{-2}\}. \tag{4.2}$$

Recall that critical points are characterized by

$$[f - F(a), h] = 0, \qquad h \in T_{F(a)} R_{l,1}\,. \tag{4.3}$$

After inserting (4.1) and (4.2) we get

$$\begin{aligned} &[f, t^\nu] \cdot 1 - [t^l(1 - xt)^{-1}, t^\nu] \cdot \alpha \\ &\quad - \sum_{\mu=0}^{l-1} [t^\mu, t^\nu] \cdot \beta_\mu = 0, \qquad \nu = 0, 1, ..., l - 1, \\ &[f, t^{l+\nu}(1 - xt)^{-1-\nu}] \cdot 1 - [t^l(1 - xt)^{-1}, t^{l+\nu}(1 - xt)^{-1-\nu}] \cdot \alpha \\ &\quad - \sum_{\mu=0}^{l-1} [t^\mu, t^{l+\nu}(1 - xt)^{-1-\nu}] \cdot \beta_\mu = 0, \qquad \nu = 0, 1. \end{aligned} \tag{4.4}$$

This homogeneous system of $l + 2$ equations for the $l + 2$ values $1, -\alpha, -\beta_0, ..., -\beta_{l-1}$ has a solution only if its determinant, call it $\psi(x)$, vanishes. Consequently, the zeros of the function ψ are of main interest.

First, observe that for each $g \in L_2[-1, +1]$, the functions

$$[g, (1 - xt)^{-\mu}], \qquad \mu = 1, 2, ...,$$

are analytic in the unit disc$\{x \in \mathbb{C}; | x | < 1\}$.

Indeed, inserting the power series

$$(1 - z)^{-\mu} = \sum_{\nu=0}^{\infty} c_\nu z^\nu, \qquad | z | < 1,$$

we obtain the representation

$$[g, (1 - xt)^{-\mu}] = \sum_{\nu=0}^{\infty} c_\nu [g, t^\nu] x^\nu. \tag{4.5}$$

Since the inner products may be estimated by applying the Cauchy–Schwarz inequality

$$|[g, t^\nu]| \leqslant \| g \| \| t^\nu \| \leqslant 2 \| g \|,$$

analyticity in the unit disk is established for the function in (4.5), the matrix elements of linear equations (4.4) are analytic functions. Since the first $l + 1$ of them define the manifold specified above, the manifold is an analytic one.

As an analytic function $\psi(x)$ is a constant or it has only isolated zeros. Now the following result is immediate.

LEMMA 4.1. *Let $f \in L_2[-1, +1]$. Then either all critical points in $R_{l,1}$ are isolated or the set of critical points is a one-dimensional analytic submanifold of $R_{l,1}$.*

We claim that the second alternative is impossible. For the special case when $l = 0$ this was already proved by Spieß using an integral transformation [16, Satz 3.6]. Our proof for arbitrary $l \geqslant 0$ is based on the following lemma; another proof is given in [11].

LEMMA 4.2. *Let $f \in L_2[-1, +1]$ and $l > 0$. Then*

$$\lim_{\substack{x \to 1 \\ x < 1}} \{[f, t^l(1 - xt)^{-1}]/\| t^l(1 - xt)^{-1} \|\} = 0.$$

Proof. There is nothing to prove if $f = 0$. Assume $f \neq 0$. Given $\epsilon > 0$, there is a $t_0 < 1$ such that

$$\int_{t_0}^{1} | f(t)|^2 \, dt < \tfrac{1}{4}\epsilon^2.$$

Furthermore, some elementary calculations establish the estimate

$$\int_{-1}^{t_0} t^{2l}(1 - xt)^{-2}\, dt < (\epsilon/4 \cdot \|f\|)^2 \int_{-1}^{1} t^{2l}(1 - xt)^{-2}\, dt$$

for x sufficiently close to 1. By splitting the integral $\int f \cdot t^l(1 - tx)^{-1}\, dt$ at $t = t_0$ and applying the Cauchy–Schwarz inequality to each term, we obtain $|[f, t^l(1 - tx)^{-1}]| < \epsilon \| t^l(1 - tx)^{-1} \|$. $\square$

Now we turn to a reduction of Eqs. (4.4) and the associated determinant. The reader may observe that the quotient space H/P_{l-1} is introduced implicitly.

Note that the approximation problem does not change if we subtract from the given function f a polynomial of degree $\leqslant l - 1$; in particular, this may be done with the best approximation of f in P_{l-1}. Thus we may assume $f \in P_{l-1}^{\perp}$.

Moreover, the representation for the elements in $R_{l,1}$ is changed. Instead of (4.1), write

$$F(a) = \alpha \cdot v(x) + \sum_{\mu=0}^{l-1} \beta_\mu t^\mu, \tag{4.6}$$

where $v(x) \in H$, $-1 < x < +1$, which is obtained from the function $t^l/(1 - xt)$ by subtracting the best approximation in P_{l-1},

$$v(x) = t^l(1 - xt)^{-1} - \sum_{\mu=0}^{l-1} [t^l(1 - tx)^{-1}, u_\mu]\, u_\mu . \tag{4.7}$$

Here, $u_0, u_1, \ldots, u_{l-1}$ are assumed to be orthogonal polynomials with norm unity which span P_{l-1}. Hence,

$$v(x) \in P_{l-1}^{\perp}, \qquad -1 < x < +1. \tag{4.8}$$

From (4.6) another basis of the tangent space is derived.

$$T_g R_{l,1} = \operatorname{span}\{1, t, \ldots, t^{l-1}, v(x), (d/dx)\, v(x)\}.$$

Let $F(a)$ be a critical point for $f \in P_{l-1}^{\perp}$. It follows from (4.3) that both $f - F(a)$ and $F(a)$ are contained in $P_{l-1}^{\perp}$. Then (4.8) implies that $F(a) = \alpha \cdot v(x)$ and that the polynomial terms in (4.6) vanish. Hence, criticality may be characterized by

$$[f - \alpha \cdot v(x), h] = 0, \qquad h \in \operatorname{span}\{v(x), (d/dx)\, v(x)\},$$

which is explicitly

$$
\begin{aligned}
[f, v(x)] \cdot 1 - [v(x), v(x)] \cdot \alpha &= 0, \\
[f, (d/dx)\, v(x)] \cdot 1 - [v(x), (d/dx)\, v(x)] \cdot \alpha &= 0.
\end{aligned}
\tag{4.9}
$$

Another possible way to derive (4.9) is to perform appropriate row and column manipulations with (4.4). Consequently, the determinant of (4.9) is a multiple of $\psi(x)$.

$$
\begin{aligned}
\psi(x) &= \text{const} \cdot \det \begin{vmatrix} [f, v(x)] & [v(x), v(x)] \\ (d/dx)[f, v(x)] & (1/2) \cdot (d/dx)[v(x), v(x)] \end{vmatrix} \\
&= \text{const} \cdot \| v(x)\|^2 \cdot (d/dx)\{[f, v(x)]/\| v(x)\|\}.
\end{aligned}
\tag{4.10}
$$

Hence, $\psi \equiv 0$ implies

$$
[f, v(x)] = c \cdot \| v(x)\|, \qquad -1 < x < 1, \tag{4.11}
$$

with some constant c independent of x. Since the orthogonal complement of $\{(1 - xt)^{-1}, -1 < x < +1\}$ in $L_2[-1, +1]$ consists only of the zero function [8], we have $c \neq 0$, apart from the trivial case that we have started with, namely, f a polynomial in P_{l-1}.

The choice $f \in P_{l-1}^{\perp}$ implies $[f, v(x)] = [f, t^l(1 - xt)^{-1}]$. By Lemma 4.2, the ratio $[f, v(x)]/\| t^l(1 - tx)^{-1} \|$ tends to zero as $x \to 1$. On the other hand, from (4.7) and Lemma 4.2 we conclude $\| v(x)\|/\| t^l(1 - tx)^{-1} \| \to 1$. This is a contradiction and we have proved

THEOREM 4.3. *For each $f \in L_2[-1, +1_4$ the critical points in $R_{l,1}$ are isolated.*

If there is an infinite number of critical points, then their characteristic numbers will have $+1$ or -1 as an accumulation point. This can be excluded for functions f with a nice behavior at the boundary points $t = +1$, $t = -1$; i.e., more explicitly, if

$$
\begin{aligned}
f(t) &= g(t) \cdot (1 - t)^{\kappa}, \qquad g(1) \neq 0,\ \kappa > 0, \\
f(t) &= h(t) \cdot (1 + t)^{\kappa}, \qquad h(-1) \neq 0,\ \kappa > 0.
\end{aligned}
$$

Here, we will restrict the explicit computations to the easiest case.

THEOREM 4.4. *If $f \in C[-1, +1]$ and $f(1) \neq 0$, $f(-1) \neq 0$, then the number of critical points in $R_{0,1}$ is finite.*

Proof. Given $\epsilon > 0$, fix $\delta > 0$ such that $|f(t) - f(t)| < \epsilon$, whenever $t > 1 - \delta$. Put $f_0 = 2 \max |f(t)|$ and estimate

$$\begin{aligned} &\left| \int_{-1}^{1} [f(t) - f(1)](dt/(1 - xt)) \right| \\ &\qquad \leqslant \int_{1-\delta}^{1} (\epsilon/(1 - xt))\, dt + \int_{-1}^{1-\delta} (f_0/(1 - xt))\, dt \\ &\qquad \leqslant (\epsilon/x) \log(1/(1 - x)) + (f_0/x) \log(2/(1 - \delta)). \end{aligned}$$

Consequently, we have

$$[f, (1 - xt)^{-1}] = f(1) \cdot \log(1/(1 - x))\{1 + o(1)\},$$

as $x \to 1$.

In the same manner the following estimates are derived.

$$\begin{aligned} [f, t(1 - xt)^{-2}] &= (1/2) f(1)(1/(1 - x))\{1 + o(1)\}, \\ [(1 - xt)^{-1}, (1 - xt)^{-1}] &= (1/2)(1/(1 - x))\{1 + o(1)\}, \\ [(1 - xt)^{-1}, t(1 - xt)^{-2}] &= (1/3)(1/(1 - x)^2)\{1 + o(1)\}. \end{aligned}$$

By insering these expressions into (4.7), we get

$$\psi(x) = (1/3) f(1) \cdot (1/(1 - x)^2) \cdot \log(1/(1 - x)) \cdot \{1 + o(1)\},$$

as $x \to 1$. A similar analysis for $x \to -1$ yields

$$\psi(x) = (1/3) f(-1)(1/(1 + x)^2) \log(1/(1 + x))\{1 + o(1)\}.$$

Hence, the zeros of ψ are contained in a compact subset of $(-1, +1)$. Since all zeros are isolated, the number is finite. $\square$

5. Discrete Case

Wolfe's result on the nonexistence of a bound refers to the approximation on intervals. It cannot be extended to the approximation on a finite point set.

Indeed, let $T = \{t_1, t_2, \ldots, t_N\}$ be a finite subset of $\mathbb{R}$, and let $C(T)$ be endowed with the inner product

$$[f, g] = \sum_{i=1}^{N} f(t_i)\, g(t_i).$$

Obviously, for each $g \in C(T)$ the inner product

$$[g, (1 - xt)^{-\nu}] \tag{5.1}$$

is a rational function in x. Its denominator is $\prod (1 - t_i x)^\nu$, and the numerator is a polynomial with degree $\leqslant (N-1)\nu$.

Now, consider the determinant of system (4.4). When applying Laplace's rule, we recognize that each term is a product of terms of the form (5.1) such that the sum of the ν's is 4. Hence, $\psi(x)$ is a rational function belonging to $R_{4(N-1),4N}$ with numerator $\prod_{i=1}^N (1 - t_i x)^4$. As an immediate consequence we have

THEOREM 5.1. *Let T be a set consisting of N points, $N \geqslant l + 3$. Then there are at most $4(N-1)$ critical points to f in $R_{l,1}$.*

(The possibility that the set of critical points form a one-dimensional manifold is excluded by the methods used in Section 4.)

6. THE CASE $r = 2$

If we apply the techniques from Section 4 to $R_{l,r}$, then we get only the information that the set of critical points may be characterized as the simultaneous zeros of r analytic functions of r variables. This means that the set of critical points is an analytic set. For $r = 2$, we obtain sharper information. At least under assumptions which are satisfied in the monospline case, the critical points are isolated or belong to one-dimensional analytic manifolds. The main idea for the treatment stems from the theory of minimal surfaces where a similar situation occurs.

For a representation of the elements of $R_{l,2}$, $l + 3$ parameters are needed. Since the functions depend linearly on $l + 1$ of them, the second derivative $d_a^2\rho$ is positive definite on an $(l+1)$-dimensional subspace. Consequently, the sum of index and nullity does not exceed 2. The following theorem provides a sharper bound.

THEOREM 6.1. *Let $g \in R_{l,2}$, $l \geqslant 1$, be a critical point to f in $R_{l,2}$. Moreover, assume that g has two distinct real poles. Then the nullity of g is at most 1.*

Proof. Write the critical point g in the form

$$g(t) = \sum_{\mu=0}^{l-2} \beta_\mu t^\mu + \sum_{\nu=1}^{2} \alpha_\nu \gamma(x_\nu, t), \tag{6.1}$$

where $x_1 < x_2$ and

$$\gamma(x, t) = (1 - xt)^{-1}. \tag{6.2}$$

This abbreviation is used not only for convenience, but also to show that the theorem is not restricted to rational functions and may be easily extended to the approximation by γ-polynomials [1, 2, 6]. Furthermore, let

$$\gamma^{(\mu)}(x, t) = (\partial^\mu/\partial x^\mu)\, \gamma(x, t), \qquad \mu = 0, 1, 2,\ldots .$$

Critically of g implies

$$\begin{aligned} [f - g, t^\mu] &= 0, & \mu &= 0, 1,\ldots, l - 2, \\ [f - g, \gamma(x_\nu, t)] &= 0, & \nu &= 0, 1, \\ [f - g, \gamma^{(1)}(x_\nu, t)] &= 0, & \nu &= 0, 1. \end{aligned}$$

When calculating second derivatives of $F(a)$, all terms vanish or are orthogonal to $f - g$ except

$$(\partial^2/\partial x_\nu^2)\, F(a) = \alpha_\nu \gamma^{(2)}(x_\nu, t), \qquad \nu = 1, 2.$$

If we associate to each $b = (\beta_0 ,\ldots, \beta_{l-2}, \xi_1, \xi_2, \eta_1, \eta_2) \in \mathbb{R}^{l+3}$ the corresponding element in the tangent space

$$h = d_a F \cdot b = \sum_{\mu=0}^{l-2} \beta_\mu t^\mu + \sum_{\nu=1}^{2} \{\xi_\nu \gamma(x_\nu, t) + \eta_\nu \alpha_\nu \gamma^{(1)}(x_\nu, t)\}, \tag{6.3}$$

then we obtain from (2.4)

$$\tfrac{1}{2} d_a^2 \rho \cdot b \cdot b = [h, h] - \sum_{\nu=1}^{2} \alpha_\nu \eta_\nu^2 [f - g, \gamma^{(2)}(x_\nu, t)]. \tag{6.4}$$

From (6.3) it is obvious that

$$d_a^2 \rho \cdot b \cdot b > 0, \qquad b \neq 0,$$

whenever $\eta_1 = \eta_2 = 0$.

Suppose that the nullity is 2. Then the kernel of $d_a^2\rho$ contains a vector b with vanishing coordinate η_2. Since the corresponding tangent vector h has the form

$$h = u(t) + \sum_{\nu=1}^{2} \xi_\nu \gamma(x_\nu, t) + \eta_1 \gamma^{(1)}(x_1, t), \tag{6.5}$$

it is contained in $R_{l+1,3}$ and has at most $l + 1$ zeros in $(-1, +1)$. Since

$$1, t,\ldots, t^{l-2}, \gamma(x_1, t), \gamma(x_2, t), \gamma^{(1)}(x_2, t) \tag{6.6}$$

is a Markov chain with $l + 2$ elements, the span contains an element $\bar{h} = d_a F \bar{b}$ which has simple zeros exactly at those points, where $h(t)$ changes its sign. After multiplying $\bar{h}$ by (-1) if necessary, we have $h(t)\,\bar{h}(t) \geqslant 0$ for all $t \in [-1, +1]$, and

$$[h, \bar{h}] > 0. \tag{6.7}$$

On the other hand, compare (6.4) and observe that h and $\bar{h}$ were constructed such that

$$\tfrac{1}{2} d_a{}^2\rho \cdot b \cdot \bar{b} = [h, \bar{h}].$$

Since $b \in \ker d_a{}^2\rho$, it follows that $[h, \bar{h}] = 0$, contradicting (6.7). □

Remark 6.2. The proof of Theorem 6.1 shows that $\ker d_a{}^2\rho$ contains only elements of the form (6.3) with $\eta_1 \neq 0$, $\eta_2 \neq 0$.

A direct generalization of Theorem 6.1 is the following. Let $g \in R_{l,r}$, $l \geqslant r - 1$, be a critical point to f in $R_{l,r}$. Moreover, assume that g has r distinct real poles. Then the nullity of g does not exceed $(r + 1)/2$. We conjecture, however, that this result is not the best possible.

As a consequence of Theorem 6.1, we obtain

THEOREM 6.3. *Each critical point in $R_{l,2}$, $l - 1$, having only distinct real poles is isolated or belongs to a one-dimensional analytic submanifold of $R_{l,2}$ which consists of critical points.*

Proof. There is nothing to prove if the nullity vanishes. Therefore, assume that the nullity is one. Referring to representation (6.1), we consider the coupled equations

$$\begin{aligned} (\partial/\partial\beta_\mu)\rho &= 0, \qquad \mu = 0, 1,\ldots, l-2, \\ (\partial/\partial\alpha_\nu)\rho &= 0, \qquad \nu = 1, 2, \\ (\partial/\partial x_1)\rho &= 0, \\ x_2 &= z. \end{aligned} \tag{6.8}$$

The Jacobian matrix for these equations is just $d_a{}^2\rho$ after replacing its last row by $(0, 0,\ldots, 0, 1)$. It follows from Remark 6.2 that this matrix is not singular. By the implicit function theorem there is a unique solution $a = a(z)$ of (6.8) in a neighborhood of the critical point g_0 whenever z is sufficiently close to $x_2^{(0)}$ the larger characteristic number of g_0. Hence, (6.8) defines a one-dimensional manifold parametrized by z. Obviously, each critical point of f in some neighborhood of g_0, must lie on the manifold. Consider the function

$$z \to a(z) \to (\partial/\partial x_2)\, \rho(a)\,|_{a=a(z)}\,.$$

This function is analytic in a neighborhood of $z = x_2^{(0)}$. It has either an isolated zero at $x_0{}^2$ or it vanishes identically. This completes the proof. □

If the given function f possesses a representation (3.8) then each critical point satisfies the assumptions of the preceding theorems. This is known from the theory of monosplines [2]. Assume that the set of critical points contains a one-dimensional manifold. Then x_2 may be taken as a coordinate. From Remark 6.2 we conclude that also $dx_1/dx_2 \neq 0$. Hence, the manifold is not a loop, and it cannot be compact. By using an open–closed argument, the manifold may be continued unless one of the characteristic numbers tends to $+1$ or -1.

Acknowledgment

The author is grateful to Henry Loeb and Richard Barrar, who invited the author to the University of Oregon and helped him to understand why the technique applied in [5] does not work in nonlinear L_2-approximation.

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